$$\left(\frac{\mathrm{d}G/\mathrm{d}\xi}{G}\right) fg' + K_G g'' = 0 \tag{2}$$

where

$$K_G = C_w \rho_\infty \mu_\infty U_\infty \left(\frac{p_e}{p_\infty}\right) \Big/ \left(\frac{G^2 \mathrm{d}\xi}{\mathrm{d}x}\right) \tag{3}$$

with $C_w = \mu_w T_\infty / \mu_\infty T_w$, the Chapman–Rubesin constant.

Constant coefficients in Eqs. (1) and (2) require[1]

$$\frac{\mathrm{d}p_e/\mathrm{d}\xi}{2p_e} = \lambda \frac{\mathrm{d}G/\mathrm{d}\xi}{G} \tag{4}$$

and

$$K_G = \frac{\mathrm{d}G/\mathrm{d}\xi}{G} \tag{5}$$

Assuming that ξ is a differentiable function of x, Eqs. (3) and (5) give p_e as

$$p_e = \frac{p_\infty}{C_w \rho_\infty \mu_\infty U_\infty} \frac{G\,\mathrm{d}G}{\mathrm{d}x} \tag{6}$$

Multiplying Eq. (4) through by $\mathrm{d}\xi/\mathrm{d}x$ and substituting for p_e from Eq. (6) give an equation for $G(x)$ as

$$GG_{xx} = (2\lambda - 1)G_x^2 \tag{7}$$

This has the solution

$$\begin{aligned} G &= (A + Bx)^{\frac{1}{2}(1-\lambda)} \qquad \lambda \neq 1 \\ &= Ae^{Bx} \qquad \lambda = 1 \end{aligned} \tag{8}$$

with A and B as constants and

$$p_e = \frac{p_\infty}{C_\infty \rho_\infty \mu_\infty U_\infty} G^{2\lambda} \tag{9}$$

All of the solutions for cases 1–4 are included within this more general solution for their particular ξ restriction.

Taking the case 2 solution as an example,

$$\xi = C_w \rho_\infty \mu_\infty U_\infty \int_0^x \left(\frac{p_e}{p_\infty}\right) \mathrm{d}x, \qquad G \approx \xi^{\frac{1}{2}}, \qquad p_e \approx \xi^N \tag{10}$$

Differentiating the relationship for ξ gives

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = C_w \rho_\infty \mu_\infty U_\infty \frac{p_e}{p_\infty} \tag{11}$$

which can be written from Eqs. (10) as

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} \approx \xi^N \tag{12}$$

That is,

$$\begin{aligned} \xi &= (A + Bx)^{1/(1-N)} \qquad N \neq 1 \\ &= e^{Ax} \qquad N = 1 \end{aligned} \tag{13}$$

Substituting this value of ξ back into Eq. (10) gives

$$\xi = (A + Bx)^{1/(1-N)}, \qquad G \approx (A + Bx)^{\frac{1}{2}(1-N)}$$

$$p_e \approx G^{2N} \qquad N \neq 1$$

$$\xi = e^{Ax}, \qquad G \approx e^{Ax/2}, \qquad p_e \approx G^2 \qquad N = 1$$

which corresponds to Eqs. (8) and (9) when $N = \lambda$, as required. Note that this case contains within it a power law and an exponential solution for the particular ξ.

References

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Reply by the Author to Jack Pike

George R. Inger*
Iowa State University, Ames, Iowa 50011-3231

PART 1 of the Comment properly draws attention to the considerations involved in tailoring at $x = x_0$ an upstream boundary layer to the new similarly solution of Ref. 1. Since space limitations there did not permit a detailed discussion of these, it will be done here. In so doing, it is deemed sufficient for engineering purposes to match the skin friction and heat transfer across x_0; a further match of the displacement thickness is only significant in the rare practical application cases[2] where strong viscous-inviscid interaction effects are important (nevertheless, a treatment of it is included for the sake of completeness).

As the Comment observes, there are several arbitrary constants associated with the G function of Ref. 1 and, hence, with the resulting pressure and boundary-layer solution properties; these, together with x_0 and both the upstream wall temperature and pressure distribution history, are thus at our disposal to effect a match in the general situation. To fix ideas, consider the specific case 3 of an exponential pressure distribution (see Ref. 1 for nomenclature). Then the upstream boundary layer at $x = x_{\bar{0}}$ must match the similarity solution values

$$\frac{p_e(x_{\bar{0}})}{p_\infty} = K_{p_1} \exp(\beta x_0 / L) \equiv K_\beta$$

with

$$C_f(x_{\bar{0}}) = \sqrt{\frac{2\beta K_\beta C_w^+}{R_{e_L}}} \cdot f_w''(\beta, g_w^+) \tag{1}$$

with

$$C_h(x_{\bar{0}}) = C_f(x_{\bar{0}}) \cdot \frac{g_w'(\beta, g_w^+)}{f_w''(\beta, g_w^+)}$$

and

$$\delta^*(x_{\bar{0}}) = (\gamma - 1) M_\infty^2 L \sqrt{\frac{C_w^+}{2\beta K_\beta R_{e_L}}} \cdot I_g(\beta, g_w^+)$$

where $g_w^+ \equiv T_w(x_0^+)/T_0$ is the wall to total temperature ratio for the downstream similarity solution and L is an arbitrary reference length to be chosen advantageously. The left sides of these expressions pertain to an arbitrary nonsimilar arbitrary pressure gradient boundary-layer solution (of which a constant pressure ramp is only a special case), which can always be described by

$$C_f(x_{\bar{0}}) = \sqrt{\frac{2[p_e(x_{\bar{0}})/p_\infty] C_w^+}{Re_{x_0}}} \cdot f_w''(x_{\bar{0}}, g_{\bar{w}}) \tag{2}$$

with

$$C_h(x_{\bar{0}}) = C_f(x_{\bar{0}}) \cdot \frac{g_w'(x_{\bar{0}}, g_{\bar{w}})}{f_w''(x_{\bar{0}}, g_{\bar{w}})}$$

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*Professor, Department of Aerospace Engineering and Engineering Mechanics, 304 Town Engineering. Associate Fellow AIAA.

and

$$\delta^*(x_{\bar{0}}) = (\sigma - 1)M_\infty^2 x_0 \sqrt{\frac{C_{\bar{w}}}{2[p_e(x_{\bar{0}})/p_\infty]Re_{x_0}}} \cdot I_g(x_{\bar{0}}, g_{\bar{w}})$$

where the $f''_w(x_{\bar{0}}, g_{\bar{w}})$, etc., here are nonsimilar values depending on the upstream history (hence on x_0) and wall temperature ratio $g_{\bar{w}} \equiv T_w(x_{\bar{0}})/T_0$ that may be different from g_w^+. Imposing the aforementioned pressure match and then combining Eqs. (1) and (2) plus their associated relations, we obtain the following pair of matching conditions for skin friction and heat transfer, respectively:

$$\beta\frac{x_0}{L} = \frac{C_{\bar{w}}}{C_w^+}\left[\frac{f''_w(x_{\bar{0}}, g_{\bar{w}})}{f''_w(\beta, g_w^+)}\right]^2 \tag{3}$$

$$\frac{g'_w(\beta, g_w^+)}{g'_w(x_{\bar{0}}, g_{\bar{w}})} = \frac{f''_w(\beta, g_w^+)}{f''_w(x_{\bar{0}}, g_{\bar{w}})} \tag{4}$$

If displacement thickness matching is also of interest, this would add a third requirement that $I_g(\beta, g_w^+)/I_g(x_{\bar{0}}, g_{\bar{w}})$ be equal to the ratio cited in Eq. (4).

To meet these requirements for chosen values of the similarity solution parameters β and g_w^+ at a given set of flight conditions (γ, M_∞, and Reynolds number per foot) we have three free parameters: L, $g_{\bar{w}}$, and the upstream length of run x_0. Choosing L so as to satisfy Eq. (3) assures an exact C_f match in the general case for any values of x_0 or $g_{\bar{w}}$; the proper choice of either x_0 or $g_{\bar{w}}$ to satisfy Eq. (4) would then match the heat transfer as well. [Note that Eq. (4) would be automatically satisfied if one adopted Reynold's analogy as an approximation. Although it is known that this analogy is not true in pressure gradients,[3] deviations from it even in $\beta \geq -0.2$ adverse pressure gradient flows are in fact not too large for highly cooled walls, and we would expect Eq. (4) to be fairly readily satisfied in practice by imposing only a slight wall temperature difference or small upstream nonsimilar pressure gradient.] Note at this juncture that this C_f match is indeed achievable even in the special case of a constant pressure ramp upstream (where the dimension x_0 is irrelevant), as is the heat transfer if we further admit a modest wall temperature discontinuity or accept the Reynolds analogy approximation for highly cooled surfaces. Going beyond this to δ^* matching, however, appears impossible for the simple ramp since I_g varies quite significantly with $-\beta$ even for highly cooled walls (Ref. 1, Fig. 7) and in a manner opposite to the sense of either f''_w or g'''_w (Ref. 1, Figs. 3 and 4). Thus δ^* matching would require an upstream pressure gradient history where x_0 is now relevant and employed to tailor the aforementioned I_g ratio.

Analogous considerations to the preceding also pertain to the case 4 type of similarity solution of Ref. 1.

We concur with the commentor's demonstration in Part 2 that the four cases given in Ref. 1 all correspond to solutions of the general G function differential equation, Eq. (1). The distinctions between cases 3 and 4 given in Ref. 1 are still deemed important, however, in discerning the basically different nature of these new solutions in the final physical variables.

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[1]Inger, G. R., "New Similarity Solutions for Hypersonic Boundary Layers with Applications to Inlet Flows," *AIAA Journal*, Vol. 33, No. 11, 1995, pp. 2080–2086.

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